Cutoff Spaces of Elliptical Gyromagnetic Planar Circuits and Waveguides Using Finite Elements

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Abstract —**The finite element method is admirably suited for the analysis of irregular planar circuits on gyromagnetic substrates. It is used in this paper to evaluate the cutoff spaces of elliptical magnetized planar circuits with electric and magnetic walls in all combinations. A knowledge of the four possible solutions and a knowledge of the rules governing the intersections of cutoff branches in the cutoff spaces are sufficient for a complete description of the related elliptical gyromagnetic waveguide with either an electric or a magnetic wall. It is also demonstrated that the demagnetized and split cutoff numbers of the resonator with magnetic sidewall and top and bottom electric walls are sufficient for the approximate description of the split phase constants of the dominant mode in the related weakly magnetized gyromagnetic waveguide problem with eitner a magnetic or an electric wall.**

I. Introduction

THE MODE nomenclature of any waveguide problem in the cutoff space is a prerequisite to understanding the propagating region. One waveguide for which the cutoff space has been fully described is the simple gyromagnetic circular waveguide [1]–[3]. A property of this type of waveguide is that a single branch in the cutoff space need not belong to a single mode over its full interval [4]–[8]. If the only interest is the mode nomenclature of the cutoff space, then the complication of having to derive the characteristic equation under propagation conditions may be replaced by the more simple task of evaluating the planar circuits obtained by terminating the two ends of the waveguide by electric or magnetic walls. There are two planar circuits associated with the waveguide with an electric sidewall and two with that of a magnetic one. This approach is employed in this paper in the study of the cutoff space of an elliptical gyromagnetic waveguide with either an electric or a magnetic sidewall. The rules governing the mode nomenclature at the intersections of the cutoff branches of the two planar circuits associated with each waveguide problem in the cutoff space have been enunciated in the related circular waveguide problem and are assumed to apply here also [4]–[8]. The cutoff numbers of the four possible planar circuits are evaluated in this paper using the finite element method. This method has now been widely employed in the study of this type of problem [9], [13]–[15], [27]. The cutoff spaces of these planar circuits differ from the corresponding circular ones in that the usual degeneracy at the origin between the orthogonal normal modes is removed by the eccentricity.

Manuscript received October 30, 1987; revised March 9, 1988.
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IEEE Log Number 8824247.

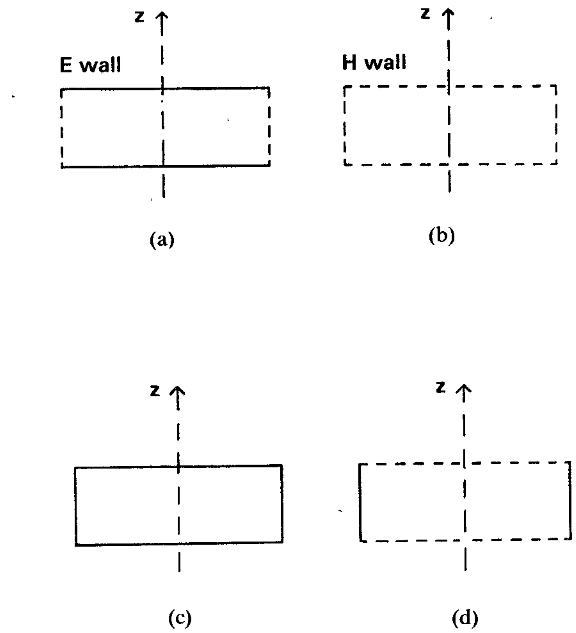

Fig. 1. Schematic diagrams of planar resonators with idealized electric or magnetic walls in all combinations.

It is separately shown, using perturbation theory, that a knowledge of the cutoff conditions of the related isotropic and gyromagnetic planar circuits with a magnetic sidewall, is sufficient to describe the normal modes in the weakly magnetized elliptical waveguide with either an electric or a magnetic wall. The approximate propagation constants obtained in this way are in fact in close agreement with those of the exact result in [21]. The normal modes are separately deduced on the basis of coupled mode theory [28]. Some measurements on elliptical planar gyromagnetic resonators are separately described. The elliptical gyromagnetic waveguide is of interest in the design of nonreciprocal gyromagnetic quarter-wave plates [25].

II. Cutoff Space of Planar Elliptical Gyromagnetic Resonators

The cutoff numbers of the four basic planar circuits associated with waveguides with electric and magnetic walls are evaluated in this paper by having recourse to the

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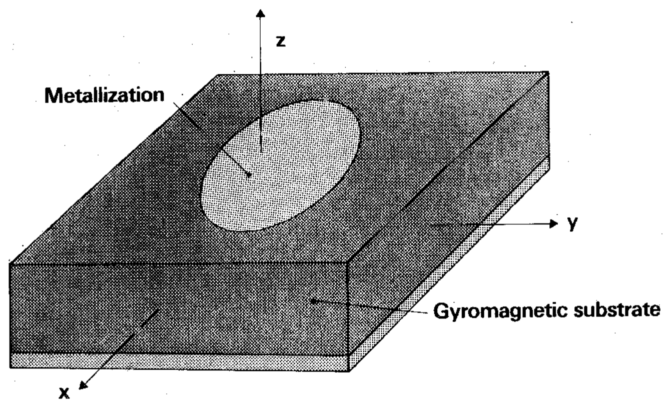

Fig. 2. Schematic diagram of planar elliptical resonator with an idealized magnetic side wall.

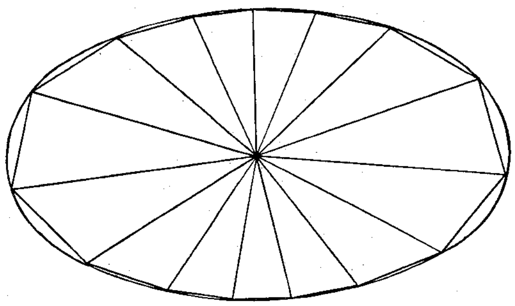

Fig. 3. Segmentation of planar elliptical resonator into 15 third-order triangular subregions.

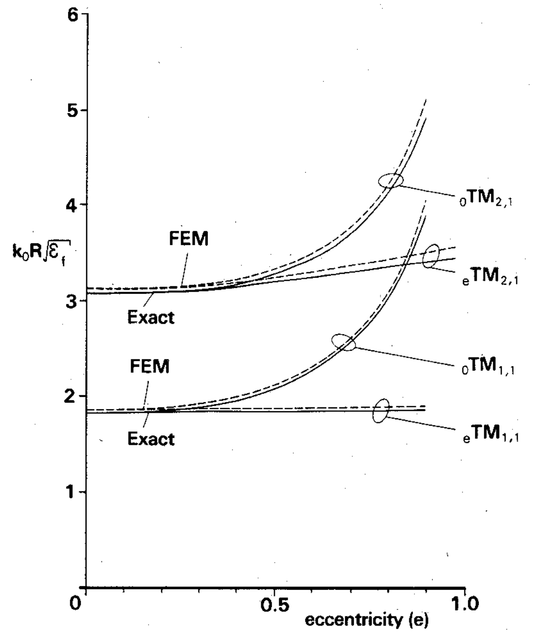

Fig. 4. Exact and FEM descriptions of the cutoff numbers of the orthogonal modes, as a function of eccentricity, in an elliptical planar resonator with idealized magnetic side walls ($_e\mathrm{TM}_{11}$, $_o\mathrm{TM}_{11}$, $_e\mathrm{TM}_{21}$, $_o\mathrm{TM}_{21}$ modes).

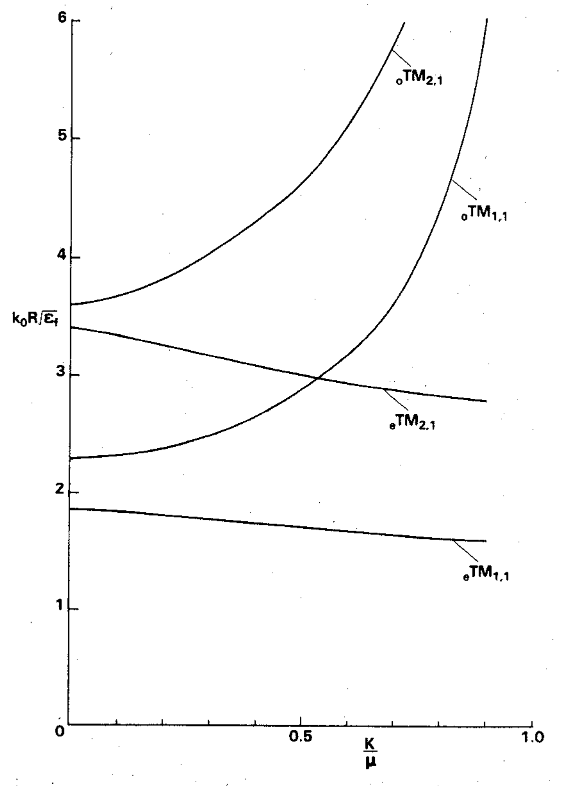

Fig. 5. Split cutoff numbers of planar elliptical gyromagnetic resonator with magnetic sidewall and electric top and bottom walls ($e = 0.60$) ($_e\mathrm{TM}_{11}$, $_o\mathrm{TM}_{11}$, $_e\mathrm{TM}_{21}$, $_o\mathrm{TM}_{21}$ modes).

finite element method. The four possibilities are illustrated in Fig. 1(a)–(d). Fig. 2 depicts, in more detail, the schematic diagram of the planar elliptical resonator with electric top and bottom walls and a magnetic sidewall. The segmentation employed in the finite element analysis is indicated in Fig. 3. It consists of 15 triangular elements within each of which a third-order polynomial approximation is employed [10], [11], [14]. The cutoff numbers of the first two pairs of orthogonal modes in this type of demagnetized elliptical resonator with a magnetic sidewall and top and bottom electric walls have been initially computed to verify the finite element software. Fig. 4 compares this result with the exact theory summarized in the Appendix [18]–[20]. An important property of this resonator is that the usual degeneracy between the orthogonal modes encountered in an isotropic disk resonator no longer exists. The mode nomenclature in such a resonator is described in terms of odd and even angular Mathieu functions Se_n and Ce_n respectively. If the order of the Mathieu function is zero then only the even mode exists. If the order is other than zero then the z component of the field patterns which correspond to the odd solution have odd symmetry with respect to the major axis of the resonator, whereas for the even solutions the corresponding field patterns have even symmetry about the major axis. Somewhat closer agreements were obtained using 32 elements and a second-order approximation within each.

Figs. 5, 6, 7, and 8 illustrate the cutoff numbers for the first few modes for the four planar circuits on a gyromagnetic substrate. The theoretical mode charts are again derived using the finite element approach. The segmenta-

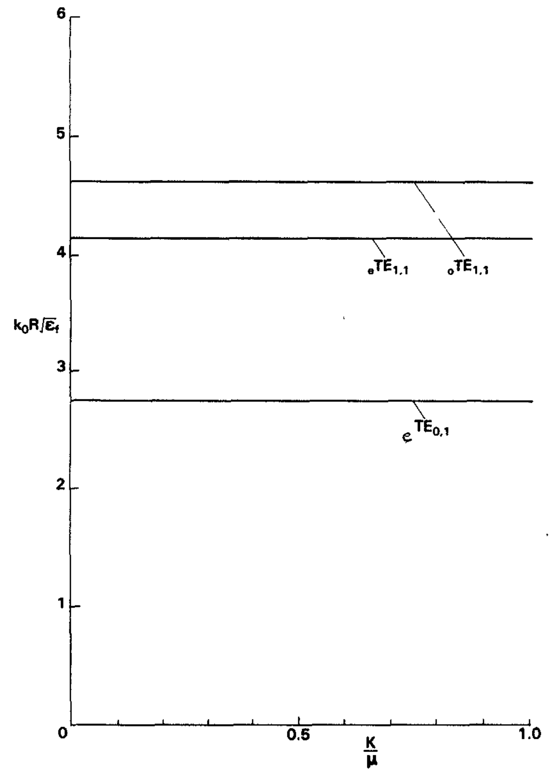

Fig. 6. Cutoff numbers of magnetized planar elliptical resonator with magnetic sidewall and magnetic top and bottom walls ($e = 0.60$) ($_e$TE$_{01}$, $_e$TE$_{11}$, $_o$TE$_{11}$ modes).

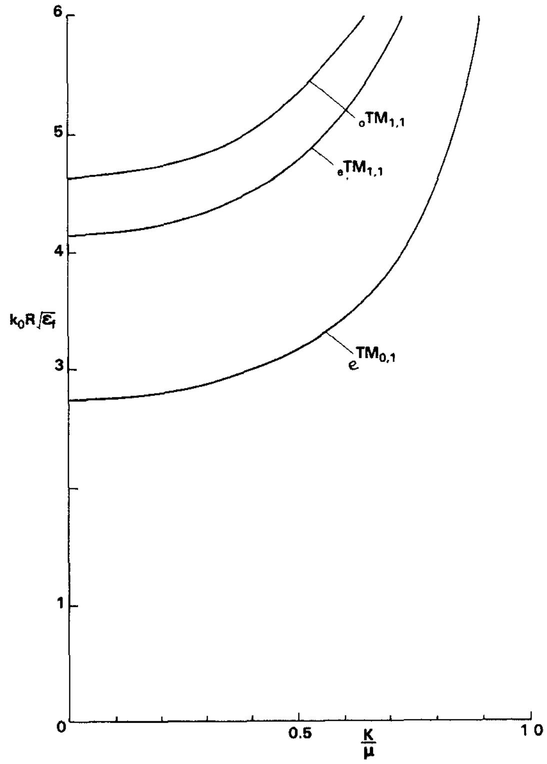

Fig. 7. Cutoff numbers of magnetized planar elliptical resonator with electric sidewall and electric top and bottom walls ($e = 0.60$) ($_e$TM$_{01}$, $_e$TM$_{11}$, $_o$TM$_{11}$ modes).

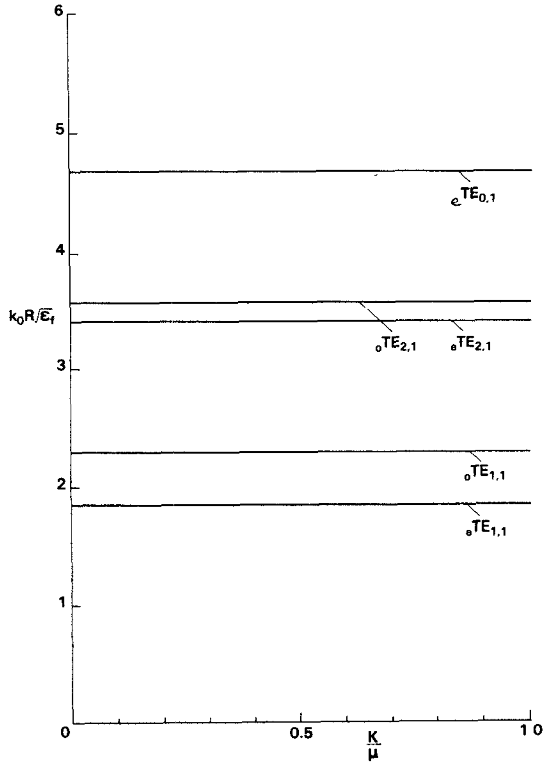

Fig. 8. Cutoff numbers of magnetized planar elliptical resonator with electric sidewall and magnetic top and bottom walls ($e = 0.60$) ($_e$TE$_{11}$, $_o$TE$_{11}$, $_e$TE$_{21}$, $_o$TE$_{21}$, $_e$TE$_{01}$ modes).

tion and polynomial approximation within each element are the same as those employed in connection with the related isotropic problem. The application of the finite element method to the solution of this class of problem is noted in [13]–[15] and [27]. It is of note that the only family of modes in a gyromagnetic resonator whose cutoff frequencies split are those which belong to the planar resonator with a magnetic sidewall and top and bottom electric walls. The planar circuit with electric end walls gives the cutoff frequencies of modes which are TM at cutoff, and the planar circuit with magnetic end walls gives the cutoff frequencies of modes which are TE at cutoff. The physical variables appearing in these illustrations are the radial wavenumbers along the major axis of the ellipse (k_0R), the diagonal and off-diagonal elements (μ and κ), respectively, of the tensor permeability, the relative dielectric constant of the ferrite or garnet substrate (ϵ_f), and the eccentricity of the resonator (e).

III. Cutoff Space of Elliptical Gyromagnetic Waveguide

The cutoff space may in general be constructed for any waveguide with either an electric or a magnetic sidewall from a knowledge of the cutoff numbers of the related planar circuits with either electric or magnetic top and bottom walls. This will now be demonstrated in the case of the elliptical gyromagnetic waveguide with an electric or magnetic wall illustrated in Fig. 9 by employing the planar solutions depicted in Figs. 5, 6, 7, and 8. Fig. 10 summarizes the cutoff space for an elliptic waveguide with an electric wall. Fig. 11 gives the same information for the

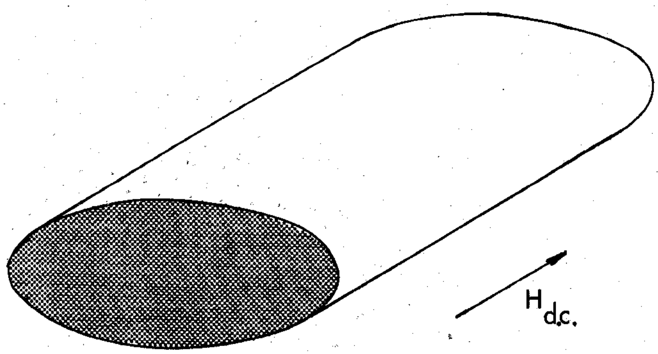

Fig. 9. Schematic diagram of elliptical gyromagnetic waveguide.

magnetic wall situation. In obtaining these results, use has been made of the rules concerning the intersections of the cutoff numbers and mode nomenclature outlined in connection with the round waveguide problem [4]–[8]. The fact that a single branch need not correspond to a single mode over its entire length is a feature in these types of waveguide. At the crossover of the ${}_e\mathrm{TE}_{01}$ and ${}_e\mathrm{TM}_{01}$ branches in the electric wall waveguide, the mode nomenclature interchanges with respect to each continuous cutoff curve. The first intersection in the magnetic wall case occurs between the ${}_o\mathrm{TM}_{11}$ and ${}_o\mathrm{TE}_{11}$ modes. Similar exchanges in the mode nomenclature occur in the cutoff space of the higher order modes not indicated in these illustrations.

IV. Experimental Mode Charts

This section describes some experimental data on the two orthogonal branches of each of the first two modes of the planar circuit with a magnetic sidewall and top and bottom electric ones. The odd and even modes are equally excited with a coupling line perpendicular to either the major or minor axes of the resonator. Fig. 12 shows a photograph of some experimental layouts. The substrate employed in this work consisted of a pure yttrium iron garnet material with a saturation magnetization of 0.1780 T and a relative dielectric constant of 15.1. The thickness of the substrate was 0.635 mm.

Fig. 13 illustrates some experimental data on the two branches of each of the first two modes of such a resonator with different eccentricities on a demagnetized garnet substrate. Fig. 14 displays the split cutoff numbers for the same two pairs of modes for one value of eccentricity ($e = 0.6$). These results are plotted against the external flux density of the magnetic circuit. Since the relationship between the internal and external direct magnetic variables in a partially magnetized magnetic geometry is not straightforward, it is possible to attempt correlation between theory and practice only at magnetic saturation of the material. The agreement between the two is adequate for engineering purposes. The garnet material employed is the same as that used in the experimental work on the isotropic circuits. Figs. 15 and 16 depict similar results but for gyromagnetic substrates with values of saturation magnetization of 0.2150 T and 0.2800 T, respectively. Similar experimental data on circuits with an eccentricity equal to 0.40 are omitted for brevity.

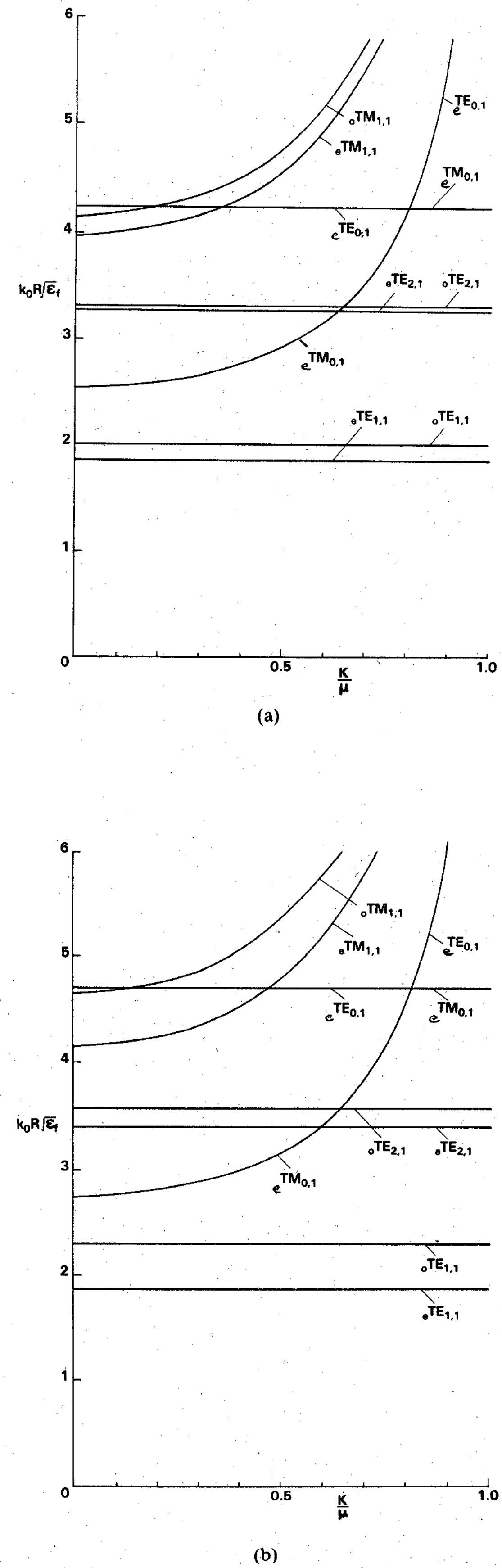

Fig. 10. (a) Cutoff space of gyromagnetic waveguide with an electric wall for the quasi-${}_e\mathrm{TE}_{11}$, ${}_o\mathrm{TE}_{11}$, ${}_e\mathrm{TM}_{01}$, ${}_e\mathrm{TE}_{21}$, ${}_o\mathrm{TE}_{21}$, ${}_e\mathrm{TM}_{11}$, ${}_o\mathrm{TM}_{11}$, and ${}_e\mathrm{TE}_{01}$ modes ($e = 0.40$). (b) Cutoff space of gyromagnetic waveguide with an electric wall for the quasi-${}_e\mathrm{TE}_{11}$, ${}_o\mathrm{TE}_{11}$, ${}_e\mathrm{TM}_{01}$, ${}_e\mathrm{TE}_{21}$, ${}_o\mathrm{TE}_{21}$, ${}_e\mathrm{TM}_{11}$, ${}_o\mathrm{TM}_{11}$, and ${}_e\mathrm{TE}_{01}$ modes ($e = 0.60$).

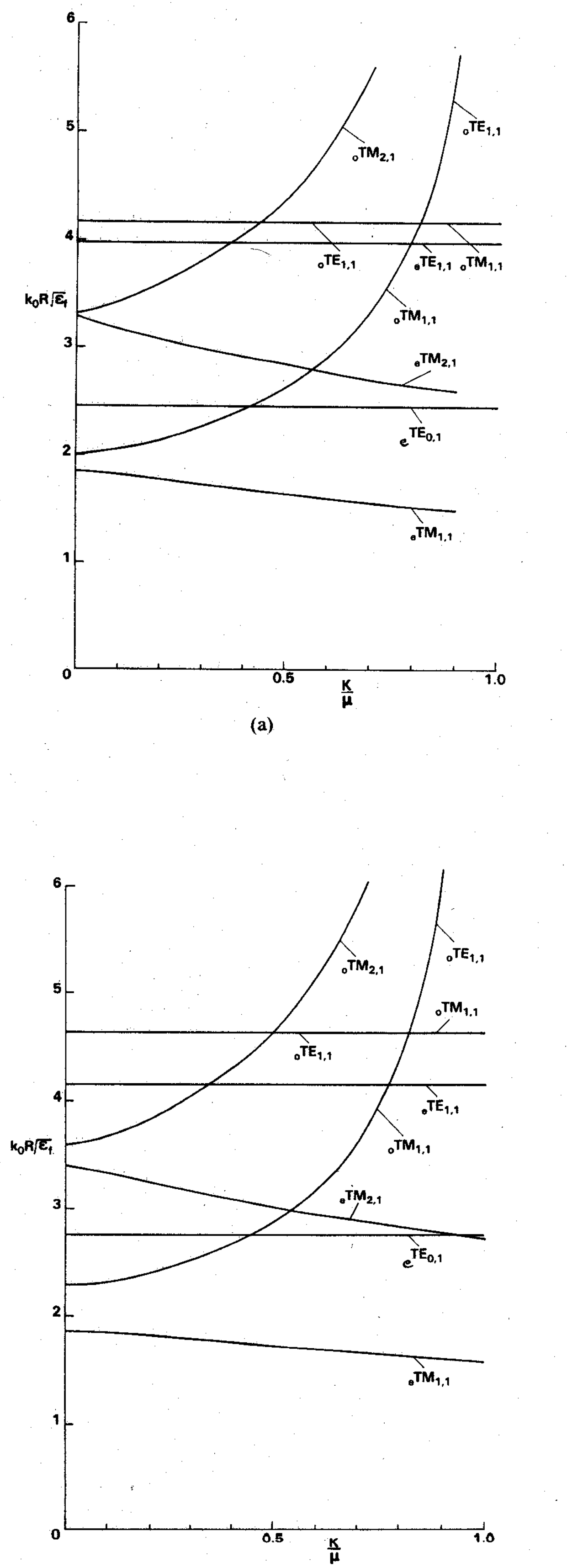

Fig. 11. Cutoff space of gyromagnetic waveguide with a magnetic wall for the quasi-$_eTM_{11}$, $_oTM_{11}$, $_eTE_{01}$, $_eTM_{21}$, $_oTM_{21}$, $_eTE_{11}$, $_oTE_{11}$ modes ($e = 0.40$). (b) Cutoff space of gyromagnetic waveguide with a magnetic wall for the quasi-$_eTM_{11}$, $_oTM_{11}$, $_eTE_{01}$, $_eTM_{21}$, $_oTM_{21}$, $_eTE_{11}$, $_oTE_{11}$ modes ($e = 0.60$).

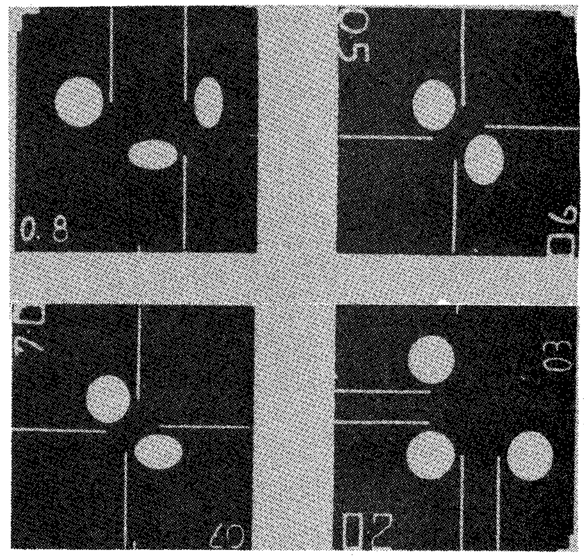

Fig. 12. Photo of planar elliptical resonators on gyromagnetic substrates.

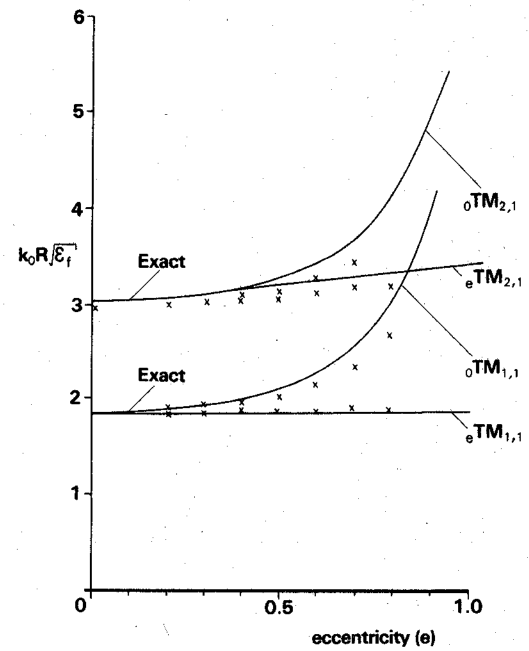

Fig. 13. Experimental frequencies of demagnetized elliptical resonator as a function of eccentricity ($_eTM_{11}$, $_oTM_{11}$, $_eTM_{21}$, $_oTM_{21}$ modes).

V. Perturbation Theory of Gyromagnetic Circular Waveguides with Electric and Magnetic Walls

Scrutiny of the dispersion relationship of a circular gyromagnetic waveguide with either an electric or a magnetic sidewall obtained using perturbation theory and the split frequencies of the associated planar circuit with a magnetic wall indicates that these may be approximately derived from a knowledge of the latter quantities. A solution of the cutoff space is therefore sufficient for the approximate formulation of the split phase constants of the dominant modes in homogeneous gyromagnetic circular waveguides with electric or magnetic wall boundary conditions. If this method is applied to the elliptical gyro-

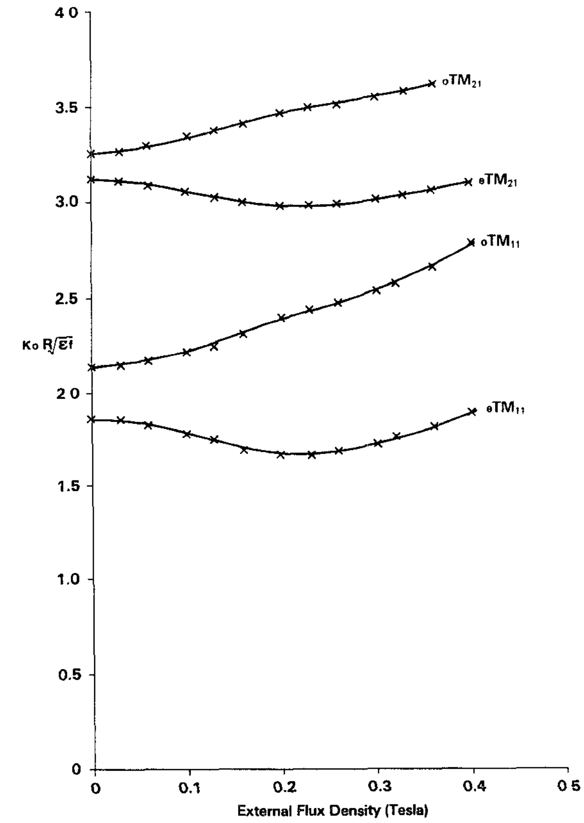

Fig. 14. Experimental split frequencies of the ${}_e\mathrm{TM}_{11}$, ${}_o\mathrm{TM}_{11}$, ${}_e\mathrm{TM}_{21}$, and ${}_o\mathrm{TM}_{21}$ modes of a magnetized planar elliptical resonator with $e = 0.60$, $\epsilon_f = 15.1$, and $M_0 = 0.1800$ T.

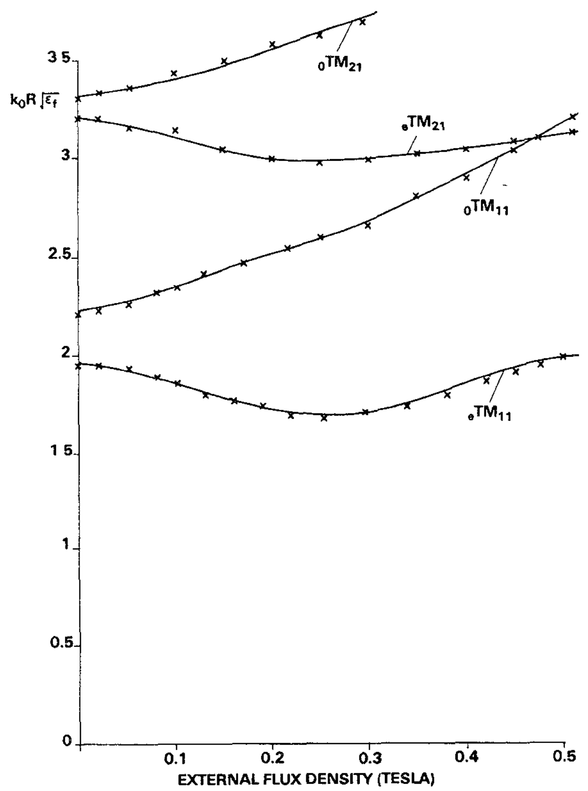

Fig. 15. Experimental split frequencies of the ${}_e\mathrm{TM}_{11}$, ${}_o\mathrm{TM}_{11}$, ${}_e\mathrm{TM}_{21}$, and ${}_o\mathrm{TM}_{21}$ modes of a magnetized planar elliptical resonator with $e = 0.60$, $\epsilon_f = 12.7$, and $M_0 = 0.2150$ T.

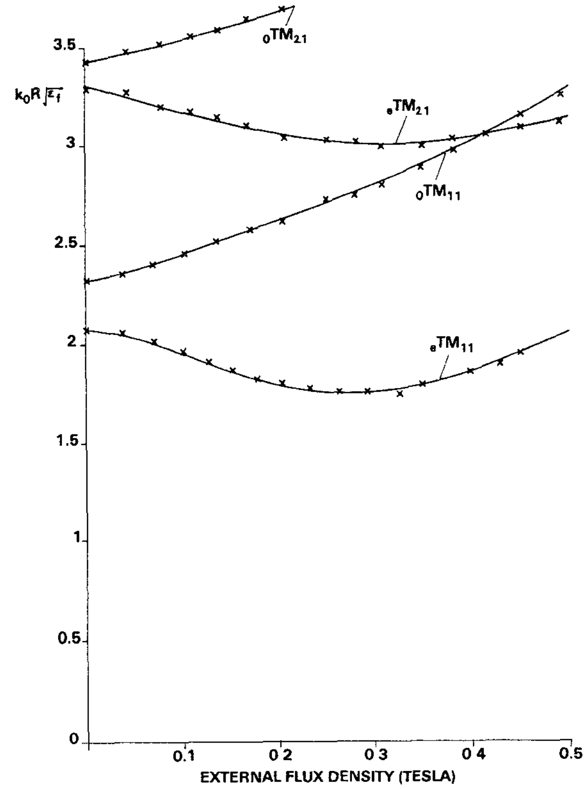

Fig. 16. Experimental split frequencies of the ${}_e\mathrm{TM}_{11}$, ${}_o\mathrm{TM}_{11}$, ${}_e\mathrm{TM}_{21}$, and ${}_o\mathrm{TM}_{21}$ modes of a magnetized planar elliptical resonator with $e = 0.60$, $\epsilon_f = 13.1$, and $M_0 = 0.2800$ T.

magnetic waveguide, then the cutoff conditions are all that is necessary to describe this class of waveguide. It is of note, however, that a single branch in the cutoff space need not describe a single mode over its entire length.

The propagation constant of a circular waveguide with either an electric or a magnetic wall for which the permeability has the tensor form

$$\begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \tag{1a}$$

may be formed from the related problem for which it has the diagonal form

$$\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \tag{1b}$$

using perturbation theory.

The required dispersion relationships for the first pair of split TE_{11} modes are then given by

$$\beta_\pm^2 \approx \beta_o^2\left(1 + C_{11}^\pm \frac{\kappa}{\mu}\right) \tag{2}$$

for the electric wall case, and

$$\beta_\pm^2 \approx \beta_o^2 + k_o^2 \epsilon_f \mu C_{11}^\pm \frac{\kappa}{\mu} \tag{3}$$

for the dominant pair of split TM_{11} modes in the magnetic wall situation. The constant $C_{11}^{\pm}$ is deduced from the perturbation development as [26], [30]

$$C_{11}^{\pm} = \frac{\pm 2}{(1.84)^2 - 1}. \tag{4}$$

The phase constant (β_0) of the related anisotropic waveguide is [8]

$$\beta_o^2 = k_o^2 \epsilon_f \mu - \left(\frac{1.84}{R}\right)^2 \frac{\mu}{\mu_z} \tag{5}$$

for the TE_{11} mode in the unperturbed circular waveguide with an electric wall and is

$$\beta_o^2 = k_o^2 \epsilon_f \mu - \left(\frac{1.84}{R}\right)^2 \tag{6}$$

for the TM_{11} mode in the unperturbed circular waveguide with a magnetic wall.

In these relations k_o is the wavenumber (rad/m), ϵ_f is the relative dielectric constant of the medium, and ϵ_0 and μ_0 are the usual constitutive parameters of free space.

The dispersion relationship of the first of these approximate solutions displays split phase constants but not split cutoff numbers, in keeping with the exact result. The dispersion relationship of the second of these solutions displays both split cutoff numbers and phase constants, again in keeping with the properties of the exact problem.

It is of note that the factor $C_{11}^{\pm}$ in the dispersion relationships coincides with the first pair of split cutoff numbers $(k_o R \sqrt{\epsilon_f \mu_{\text{eff}}})_{\pm 1,1}$ about the cutoff number at the origin, $(k_o R \sqrt{\epsilon_f \mu})_{1,1}$:

$$2\left[\frac{\left(k_o R \sqrt{\epsilon_f \mu}\right)_{1,1} - \left(k_o R \sqrt{\epsilon_f \mu_{\text{eff}}}\right)_{\pm 1,1}}{\left(k_o R \sqrt{\epsilon_f \mu}\right)_{1,1}}\right] \approx C_{11}^{\pm} \frac{\kappa}{\mu} \tag{7}$$

of the characteristic equation of a weakly magnetized planar disk resonator with a magnetic wall [29]

$$J_n'(k_e R) - n\left(\frac{\kappa}{\mu}\right)\frac{J_n(k_e R)}{k_e R} = 0 \tag{8}$$

provided

$$k_e R = k_o R \sqrt{\epsilon_f \mu_{\text{eff}}} \tag{9}$$

and

$$\mu_{\text{eff}} = \frac{\mu^2 - \kappa^2}{\mu} \approx \frac{\mu^2 - \left(C_{11}^{\pm}\kappa\right)^2}{\mu}. \tag{10}$$

Scrutiny of the approximate dispersion relationships in this section and the exact ones using the appropriate characteristic equations in [1]–[3] and [8] indicates that the former descriptions are adequate for everyday engineering provided

$$0 \leqslant \frac{\kappa}{\mu} \leqslant 0.5.$$

A similar development for a square waveguide gives $C_{11}^{\pm} = \pm 0.81$ [28].

VI. Approximate Split Phase Constants of Circular Gyromagnetic Waveguide Using the Finite Element Method

The coefficients $C_{11}^{\pm}$ appearing both in the perturbation formulation of the phase constants and in the description of the weakly magnetized planar resonator assume that the fields of the gyromagnetic waveguide are equal to those of the isotropic waveguide. This assumption breaks down, in each instance, for the lower of the two split branches. In the magnetic wall case this may be understood by recognizing that the fields of the exact solution are cutoff whereas these are assumed propagating in the perturbation formulation.

A more accurate relationship for this quantity may be empirically constructed using the finite element approach by investigating the condition

$$\beta_{\mp} = 0 \tag{11}$$

in the magnetic wall problem. This gives $C_{11}^{\pm}$ in terms of the cutoff frequencies $(k_o R \sqrt{\epsilon_f \mu})_{\pm 1,1}$:

$$k_o^2 \epsilon_f \mu \left(1 + C_{11}^{\pm}\frac{\kappa}{\mu}\right) - \left(\frac{(k_e R)_{1,1}}{R}\right)^2 = 0 \tag{12}$$

instead of the relationships in (4). The quantity $(k_e R)_{1,1}$ refers to the demagnetized cutoff number of the dominant mode.

Substituting this quantity in (12) into the equation for $\beta_{\pm}$ leads in the electric wall case to

$$\frac{\beta_{\pm}}{k_0} = \frac{(k_e R)_{1,1}}{\left(k_o R \sqrt{\epsilon_f \mu}\right)_{\pm 1,1}}\left(\frac{\beta_o}{k_o}\right) \tag{13}$$

and in the magnetic wall case to

$$\frac{\beta_{\mp}}{k_o} = \sqrt{\left\{\left[\frac{(k_e R)_{1,1}}{\left(k_o R \sqrt{\epsilon_f \mu}\right)_{\pm 1,1}}\right]^2 \epsilon_f \mu - \left(\frac{(k_e R)_{1,1}}{k_o R}\right)^2\right\}}. \tag{14}$$

These relationships are adequate for engineering purposes in the case of the dominant modes in the range

$$0 \leqslant \frac{\kappa}{\mu} \leqslant 1.$$

Figs. 17 and 18 compare the accuracy of this approximation to that of perturbation theory for both the electric and magnetic wall waveguides.

VII. Approximate Split Phase Constants of Elliptical Gyromagnetic Waveguide Using Finite Elements

If the relationship between the first dominant elliptically polarized split phase constants and cutoff numbers is extended to the elliptical waveguide shown in Fig. 9, then the solution of the planar circuit described here provides an approximate solution to the elliptical waveguide also.

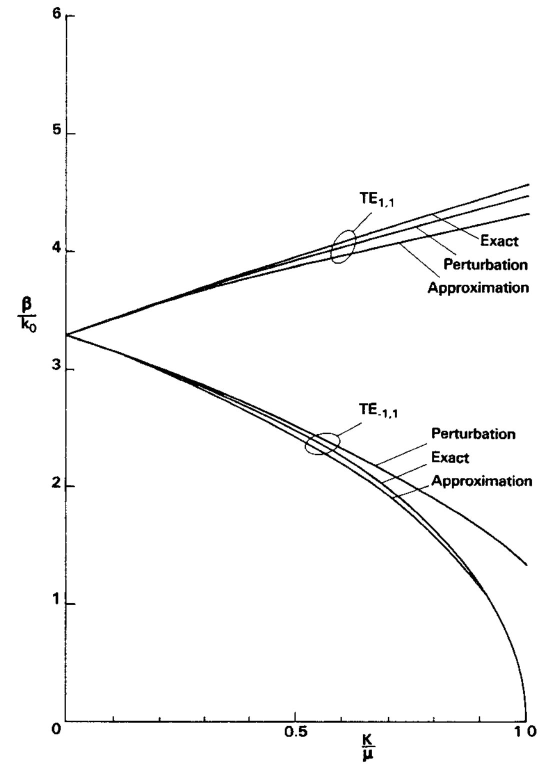

Fig. 17. Comparison of the exact, perturbation, and approximation formulations of the split phase constants of the circular gyromagnetic waveguide with an electric wall (TE_{11} mode: $k_oR = 0.9$, $\epsilon_f = 15$, $\mu = \mu_z = 1$).

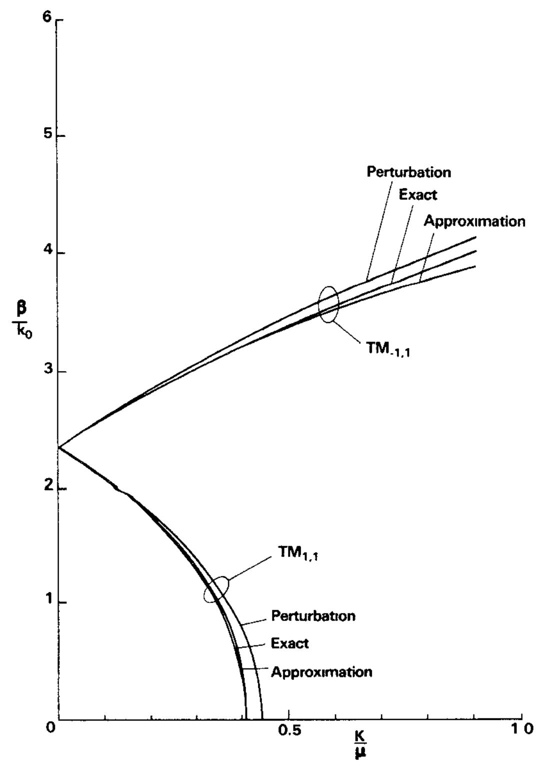

Fig. 18. Comparison of the exact, perturbation and approximation formulations of the split phase constants of the circular gyromagnetic waveguide with a magnetic wall (TM_{11} mode: $k_oR = 0.6$, $\epsilon_f = 15$, $\mu = \mu_z = 1$).

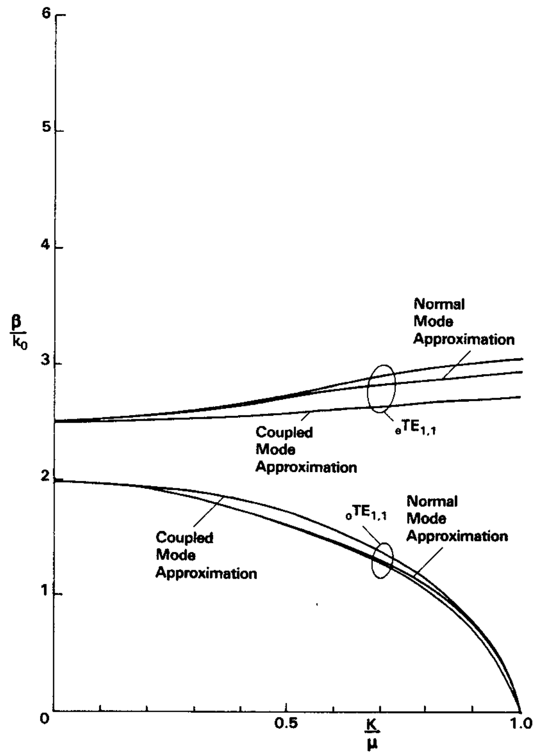

Fig. 19. Split phase constants of elliptical gyromagnetic waveguide with an electric wall using exact, approximate normal mode, and coupled mode theory (${}_eTE_{11}$ and ${}_oTE_{11}$ modes: $e = 0.648$, $\epsilon_f = 10$, $k_oR = 0.9696$).

An exact solution to the elliptical gyromagnetic waveguide with an electric wall is available in the literature [21], and this gives an opportunity to test this assumption. Specializing equation (13) for this problem gives

$$\frac{\beta_\pm}{k_o} = \frac{(k_eR)_{e,o}}{\left(k_oR\sqrt{\epsilon_f\mu}\right)_{e,o}} \frac{\beta_{e,o}}{k_o} \tag{15}$$

where

$$\frac{\beta_{e,o}}{k_o} = \sqrt{\left\{\epsilon_f\mu - \left[\frac{(k_eR)_{e,o}}{k_oR}\right]^2\left(\frac{\mu}{\mu_z}\right)\right\}}. \tag{16}$$

The corresponding approximation for the magnetic wall waveguide with the aid of (14) is written as

$$\frac{\beta_\mp}{k_o} = \sqrt{\left\{\left[\frac{(k_eR)_{e,o}}{\left(k_oR\sqrt{\epsilon_f\mu}\right)_{e,o}}\right]^2 \epsilon_f\mu - \left[\frac{(k_eR)_{e,o}}{k_oR}\right]^2\right\}}. \tag{17}$$

In the preceding equations $(k_eR)_{e,o}$ refers to the roots of the demagnetized circuit and $(k_oR\sqrt{\epsilon_f\mu})_{e,o}$ are the appropriate values at κ/μ. Figs. 19 and 20 compare the exact and approximate results for two typical arrangements. Scrutiny of these illustrations indicates that this approximation appears to be valid for the interval

$$0 \leqslant \frac{\kappa}{\mu} \leqslant 1.0.$$

If the elliptical waveguide is excited simultaneously by

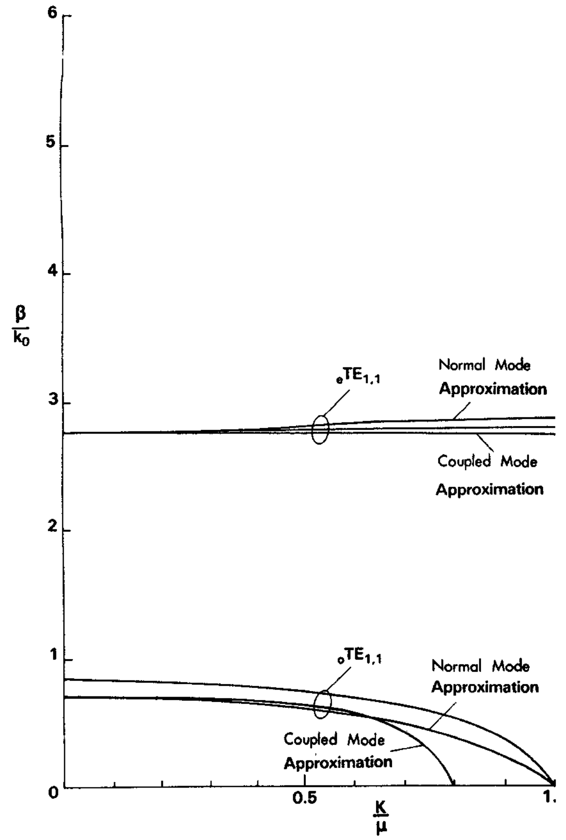

Fig. 20. Split phase constants of elliptical gyromagnetic waveguide with an electric wall using exact and approximate normal mode and coupled mode theory (${}_e\mathrm{TE}_{11}$ and ${}_o\mathrm{TE}_{11}$ modes: $e=0.887$, $\epsilon_f=10$, $k_oR=1.24$).

both normal modes then the ensuing elliptically polarized wave will exhibit the classic Faraday rotation effect, as is well understood. The possibility of constructing quarter-wave plates using this waveguide is noted [25], [28].

VIII. Coupled Wave Theory of the Normal Modes

Coupled waveguide problems are best discussed in terms of coupled modes and normal modes. If the degeneracy between either polarization is somehow removed, as in a ferrite medium, then the variables whose degeneracy is removed are known as the normal modes of the structure and the other variables are known as the coupled ones. The coupled modes, in this instance, are the orthogonal odd- and even-mode solutions of the problem. The polarizations of the normal modes reduce to those of the two counterrotating elliptical polarizations of the demagnetized waveguide. The coupled mode description may be summarized by

$$\beta_{\pm}^2=\left(\frac{\beta_e^2+\beta_o^2}{2}\right)\pm\sqrt{\left[\left(\frac{\beta_e^2\pm\beta_o^2}{2}\right)^2+k^2\right]}. \qquad (18)$$

It is of note that the phase constants of the normal modes are degenerate to those of the orthogonal or coupled ones. It is readily demonstrated that the coupling coefficient (k) in an infinite medium is given by [28].

$$k=k_o^2\epsilon_f\kappa. \qquad (19)$$

In the waveguide problem considered here it is empirically constructed in terms of the phase constants of the two coupled modes and the arbitrary constant $C_{11}^{\pm}$ encountered in connection with the approximate normal mode problem, on the basis of a good fit with the exact result as

$$k=C_{11}^{\pm}\left(\frac{\beta_e+\beta_o}{2}\right)^2\kappa. \qquad (20)$$

The result obtained in this manner is superimposed on Figs. 19 and 20.

IX. Conclusions

The finite element method has been employed to calculate the cutoff numbers of a planar gyromagnetic resonator with electric and magnetic walls in all combinations. The solutions to these four problems are all that is necessary to fully describe the cutoff space of the related elliptical waveguide with either an electric or a magnetic wall. Approximate formulations of the split phase constants of these types of gyromagnetic waveguides are also derived. The solution associated with the electric wall problem is in keeping with that of the exact problem.

Appendix

The exact solution to the isotropic elliptical planar resonator with top and bottom electric walls and a magnetic sidewall is given below in terms of the derivatives (Ce_m', Se_m') of odd and even radial Mathieu functions of the first kind (Ce_m, Se_m) [19], [20], [22]–[24]:

$$\mathrm{Ce}_m'(\xi_0,q)=0 \quad \text{for even modes} \qquad (A1)$$

$$\mathrm{Se}_m'(\xi_0,q)=0 \quad \text{for odd modes.} \qquad (A2)$$

The quantity ξ_0 defines the boundary of the resonator and is related to the eccentricity e by

$$\xi_0=\cosh^{-1}\left(\frac{1}{e}\right) \qquad (A3a)$$

and q is given by

$$q=\frac{1}{4}e^2\omega^2\mu_0\mu_r\epsilon_0\epsilon_rR^2. \qquad (A3b)$$

R is the radius along the major axis of the circuit and the other quantities have the usual meaning.

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